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Approximation by Incomplete Polynomials

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For any Θ with $0 < \Theta < 1$, it is known that the set of all incomplete polynomials of form

$$P_n(x) = \sum_{k=\mu}^{n} a_k x^k, \qquad \mu \geqslant \Theta \cdot n \tag{1}$$

is not dense in $C_0[a, 1] := \{f \in C[a, 1] : f(a) = 0\}$ if $a < \Theta^2$. In this paper, we prove that the set (1) of incomplete polynomials is dense in $C_0[a, 1]$ if $a \geqslant \Theta^2$ and even has the Jackson property on $[a, 1]$ if $a > \Theta^2$.

1. INTRODUCTION

Lorentz [5, 6] introduced the term "incomplete polynomials" to denote polynomials of the form (1) and defined the functions $\Delta(\Theta)$ and $\delta(\Theta)$ to describe the approximation properties of incomplete polynomials.

DEFINITION 1. For each Θ with $0 < \Theta < 1, \Delta(\Theta)$ is defined by the following property. If $\{P_n\}$ is a sequence of polynomials (defined for infinitely many n) of the form (1) and if

$$\| P_n \| := \max_{0 \leqslant x \leqslant 1} | P_n(x)| \leqslant 1,$$

then $P_n(x) \to 0$ uniformly on each interval $[0, d]$, $d < \Delta(\Theta)$; but this is not always true for $d > \Delta(\Theta)$.

DEFINITION 2. For each Θ with $0 < \Theta < 1$, $\delta = \delta(\Theta) > 0$ is the smallest positive number with the following property. If $f(x)$ is continuous on $[\delta, 1]$, then there exists a sequence $\{P_n(x)\}_{n=1}^{\infty}$ of polynomials of the form (1) which converges uniformly to $f(x)$ on all compact subsets of $(\delta, 1]$.

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It is known (see Lorentz [5], Lorentz and Kemperman [4]) that

$$\Theta^2 \leqslant \Delta(\Theta) < \Theta, \Theta^2 \leqslant \delta(\Theta) \leqslant \Theta \qquad \text{for } 0 < \Theta < 1. \tag{2}$$

In Section 2, Theorem 1, we show that $\Delta(\Theta) \leqslant \Theta^2$ and $\delta(\Theta) \leqslant \Theta^2$ and therefore

$$\Delta(\Theta) = \delta(\Theta) = \Theta^2 \qquad \text{for} \quad 0 < \Theta < 1. \tag{3}$$

Professors Saff and Varga were so kind to inform me that they also have proved these results earlier and in a different way. (See [7, 8].)

In Section 3, Theorem 2, we recall a general result on "incomplete Λ-polynomials", which was published in 1976 and 1977 (see v. Golitschek [2, Satz 2; 3, Theorem 1]). For the special case of algebraic polynomials, Theorem 2 immediately leads to the inequality $\delta(\Theta) \leqslant \Theta^2$ and even to asymptotically best possible rates of convergence for incomplete polynomials.

2. Density

We state our first main result.

THEOREM 1. *For any Θ with $0 < \Theta < 1$ and any function $f \in C[0, 1]$ with $f(x) = 0$, $0 \leqslant x \leqslant \Theta^2$, there exists a sequence $\{P_n(x)\}_{n=1}^{\infty}$ of polynomials of form* (1) *such that*

$$\lim_{n\to\infty} P_n(x) = f(x) \tag{4}$$

uniformly on [0, 1]. *Hence, the inequalities $\Delta(\Theta) \leqslant \Theta^2$, $\delta(\Theta) \leqslant \Theta^2$ and, by* (2), *the equalities* (3) *hold.*

Proof. Let $\epsilon > 0$. We choose $\eta = \eta(\epsilon) > 0$ so small that $a := \Theta^2 + \eta < 1$ and that the function $g \in C[0, 1]$,

$$\begin{aligned} g(x) &:= f(x - \eta), \qquad & a \leqslant x \leqslant 1, \\ &:= \quad 0, \qquad & 0 \leqslant x \leqslant a, \end{aligned}$$

satisfies $\| f - g \| < \epsilon$. Then we take an integer $M = M(\epsilon)$ so large that

$$4e\Theta^{2M}a^{-M} \leqslant 1$$

and an integer $m = m(\epsilon)$ so large that

$$\frac{5}{4}\, w(G; m^{-1/2}) + \| f \|\, e^2 2^{1-ma^M} \leqslant \epsilon,$$

where $G \in C[0, 1]$ is defined by $G(x) := g(x^{1/M})$ and $w(G; \cdot)$ denotes the modulus of continuity of G. Since $G(x) = 0$ for $0 \leqslant x \leqslant a^M$, its Bernstein polynomial of degree m has the form

$$B_m(G; x) := \sum_{j=0}^{m} G\left(\frac{j}{m}\right)\binom{m}{j} x^j(1-x)^{m-j} = \sum_{q=0}^{m} a_{qm} x^q,$$

where $a_{qm} = 0$ for $0 \leqslant q \leqslant ma^M$ and

$$|a_{qm}| = \left| \sum_{ma^M < j \leqslant q} G\left(\frac{j}{m}\right)\binom{m}{j}\binom{m-j}{m-q}(-1)^{q-j} \right| \leqslant \|G\| \binom{m}{q} 2^q$$
$$\leqslant \|f\| (2e)^q a^{-Mq} \qquad \text{for} \quad ma^M < q \leqslant m.$$

It is known that

$$\|g(x) - B_m(G; x^M)\| = \|G(x) - B_m(G; x)\| \leqslant \tfrac{5}{4} w(G; m^{-1/2}).$$

We now consider any integer n with $n > Mm/\Theta$ and replace in

$$B_m(G; x^M) = \sum_{ma^M < q \leqslant m} a_{qm} x^{Mq}$$

each monomial x^{Mq} by a suitable incomplete polynomial $Q_{q,n}$ of degree n and form (1), for which (see [1, Lemma 2])

$$A_{qn} := \|x^{Mq} - Q_{q,n}(x)\| \leqslant \prod_{\Theta \cdot n \leqslant k \leqslant n} \frac{k - Mq}{k + Mq}$$
$$\leqslant \exp\left\{-2Mq \sum_{\Theta \cdot n \leqslant k \leqslant n} 1/k\right\},$$

where we have applied the inequality $(1-t)/(1+t) \leqslant e^{-2t}$ factorwise for $t = Mq/k$. Hence,

$$A_{qn} \leqslant \exp\left\{-2Mq \log \frac{n}{1 + \Theta \cdot n}\right\} \leqslant e^2 \Theta^{2Mq}, \qquad ma^M < q \leqslant m.$$

The polynomial $P_n(x) := \sum_{ma^M < q \leqslant m} a_{qm} Q_{q,n}(x)$ is of form (1) and

$$\|g - P_n\| \leqslant \|g(x) - B_m(G; x^M)\| + \sum_{ma^M < q \leqslant m} |a_{qm}| A_{qn}.$$

Since

$$|a_{qm}| A_{qn} \leqslant \|f\| (2e)^q a^{-Mq} e^2 \Theta^{2Mq} \leqslant \|f\| e^2 2^{-q},$$

we are led to

$$\|g - P_n\| \leqslant \epsilon \text{ and } \|f - P_n\| < 2\epsilon.$$

Hence, as $\epsilon \to 0$, we can choose a sequence $\{P_n\}$ of form (1), which converges to $f(x)$ uniformly on [0, 1]. That concludes the proof of Theorem 1.

3. Rate of Convergence

Throughout Section 3 we use the following notations. r is a nonnegative integer, a and p are real numbers with $0 < a < 1$, and $1 \leqslant p < \infty$,

$$X_\infty{}^0[a, 1] := C[a, 1],\ X_\infty{}^r[a, 1] := \{f \in C[a, 1] : f^{(r)} \in C[a, 1]\},$$
$$X_p{}^0[a, 1] := L^p[a, 1],\ X_p{}^r[a, 1] := \{f : f^{(r-1)} \text{ absolutely continuous on } [a, 1], f^{(r)} \in L^p[a, 1]\}.$$

For $g \in X_p{}^0[a, 1]$, we define the norm

$$\| g \|_{\infty,a,1} := \max_{a \leqslant x \leqslant 1} | g(x) | ,$$

$$\| g \|_{p,a,1} := \left(\int_a^1 | g(x) |^p \, dx \right)^{1/p}, \qquad \text{if} \quad 1 \leqslant p < \infty,$$

and the L^p modulus of continuity $w_p(g; \cdot)$, $1 \leqslant p \leqslant \infty$, by

$$w_p(g; h) := \sup_{|t| \leqslant h} \| g(x + t) - g(x) \|_{p,a,1}, \qquad 0 \leqslant h \leqslant 1 - a,$$

where we continue the function g outside of $[a, 1]$ by

$$\begin{aligned} g(x) &:= g(2a - x), \qquad & 2a - 1 \leqslant x \leqslant a, \\ &:= g(2 - x), \qquad & 1 \leqslant x \leqslant 2 - a. \end{aligned}$$

Finally, $\Lambda = \{\lambda_k\}_{k=1}^\infty$ is any sequence of distinct complex numbers with positive real parts such that

$$\sum_{k=1}^\infty \operatorname{Re} \lambda_k / | \lambda_k |^2 = \infty \tag{5}$$

and

$$0 < | \lambda_k | \leqslant | \lambda_{k+1} |, \ | \lambda_i - \lambda_k | \geqslant M(i - k) \qquad \text{for all } k_0 \leqslant k < i, \tag{6}$$

where M is a positive constant and k_0 is a positive integer. In [3, Theorem 1] we have proved the following result.

THEOREM 2. *For $1 \leqslant p \leqslant \infty$ and any $\epsilon > 0$ there exist positive numbers K and c (only depending on r, p, a, M, and ϵ) with the following property. For any $f \in X_p{}^r[a, 1]$ and any sufficiently large integer s we can find coefficients c_{ks}, $\psi_s \leqslant k \leqslant s$, such that the inequality*

$$\left\| f(x) - \sum_{k=\Psi_s}^{s} c_{ks} x^{\lambda_k} \right\|_{p,a,1} \leqslant K \Psi_s^{-r} w_p(f^{(r)}; \Psi_s^{-1}) + O(e^{-c\Psi_s}) \tag{7}$$

holds. The integer ψ_s depends on a, Λ, ϵ and is to be the largest integer for which

$$\sum_{k=\Psi_s}^{s} \operatorname{Re} \lambda_k / | \lambda_k |^2 \geqslant \epsilon - \tfrac{1}{2} \log a \tag{8}$$

is satisfied.

Remark. It follows from the proof of [2, Satz 2] that our above Theorem 2 is still valid, if we replace the conditions (6) by

$$| \lambda_k | \geqslant Mk \qquad \text{for all } k \geqslant k_0 . \tag{6$'$}$$

Our next theorem is an immediate corollary of Theorem 2 for the special sequence $\Lambda = \{k\}_{k=1}^{\infty}$ of positive integers. It states that the incomplete polynomials of form (1) have the Jackson property on $[a, 1]$ if $a > \Theta^2$.

THEOREM 3. *For $1 \leqslant p \leqslant \infty$, $0 < \Theta < 1$, and any number a with $a > \Theta^2$ there exist positive numbers K^* and c^* (only depending on r, p, a, Θ) with the following property. For any function $f \in X_p{}^r[a, 1]$ and any sufficiently large integer n we can find algebraic polynomials P_n of form (1) such that the inequality*

$$\| f - P_n \|_{p,a,1} \leqslant K^* n^{-r} w_p(f^{(r)}; n^{-1}) + O(e^{-c^* n}) \tag{9}$$

holds.

Proof. We define $\eta > 0$ by $a = \theta^2 e^{2\eta}$ and $\epsilon := \eta/2$. If $n \geqslant 2/(\eta \cdot \Theta)$, we obtain from the definition of Ψ_n that

$$-1/\Psi_n + \log \frac{n}{\Psi_n} \leqslant \sum_{k=1+\Psi_n}^{n} 1/k < \epsilon - \tfrac{1}{2} \log a = -\log \Theta - \eta/2.$$

Hence,

$$\Psi_n e^{1/\Psi_n} > n \cdot \Theta \cdot e^{\eta/2} \geqslant n \cdot \Theta \cdot e^{1/(n \cdot \Theta)}$$

and

$$\Psi_n > \Theta \cdot n.$$

Therefore, the application of Theorem 2 leads to the statement of Theorem 3.

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